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On the Automodel Two-Dimensional and Axisymmetric Steady Motion of a Gas,

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The equations of steady axisymmetrical flow may be written

$$\left.\begin{aligned}
&u\frac{\partial u}{\partial r}+\frac{v}{r}\frac{\partial u}{\partial \theta}-\frac{v^2}{r}+\frac{1}{\rho}\frac{\partial p}{\partial r}=0\\
&u\frac{\partial v}{\partial r}+\frac{v}{r}\frac{\partial v}{\partial \theta}+\frac{uv}{r}+\frac{1}{r\rho}\frac{\partial p}{\partial \theta}=0\\
&r\frac{\partial}{\partial r}(\rho u)+\rho u+\frac{\partial}{\partial \theta}(\rho v)+\rho(u+v\cot\theta)=0\\
&ru\frac{\partial s}{\partial r}+v\frac{\partial s}{\partial \theta}=0
\end{aligned}\right\}\quad(1)$$

Here r is a radius vector; θ is the angle between any axis (coinciding with the undisturbed motion of the gas) and the radius vector; u is the projection of the velocity on r; v is the projection of the velocity on a perpendicular to r; ρ is the density; p is pressure; $s = p\rho^{-k}$ is a quantity characterizing the entropy of the gas. The term $u+v\cot\theta$ characterizes the axisymmetricalness of the stream in the case of two-dimensional flow this term vanishes.

Let us introduce the quantity $w = \frac{p}{\rho}$; then (1) becomes

$$\left.\begin{aligned}
&ruu'_r+vu'_\theta-v^2+rw'_r+rw\frac{\rho'_r}{\rho}=0\\
&ruv'_r+vv'_\theta+uv+w'_\theta+w\frac{\rho'_\theta}{\rho}=0\\
&r\,u'_r+u+v'_\theta+vu\frac{\rho'_r}{\rho}+v\frac{\rho'_\theta}{\rho}+u+v\cot\theta=0\\
&ru\frac{w'_r}{w}+v\frac{w'_\theta}{w}=(k-1)\left(ru\frac{\rho'_r}{\rho}+v\frac{\rho'_\theta}{\rho}\right)
\end{aligned}\right\}\quad(2)$$

Let us introduce

$$u = r^{\alpha_1} x_1,\ v = r^{\alpha_1} x_2,\ w = r^{2\alpha_1} y,\ \rho = r^{\alpha_2}\eta \tag{3}$$

where x_1, x_2, y, η are functions of one independent variable

$$z = r^{\alpha_3} e^{\theta} \tag{4}$$

As a result, we arrive at a system of equations, describing the automodel steady motion of a gas:

$$\left.\begin{aligned}
&\alpha_1 x_1^2 + \alpha_3 x_1 x_1' + x_2 x_1' - x_2^2 + (2\alpha_1 + \alpha_2 + \alpha_3 \frac{\eta'}{\eta}) y + \alpha_3 y' = 0) \\
&(1 + \alpha_1) x_1 x_2 + \alpha_3 x_1 x_2' + x_2 x_1' + y \frac{\eta'}{\eta} + y' = 0 \\
&x_1(1+\alpha_1 + \alpha_2) + \alpha_3 x_1' + x_2' + \frac{\eta'}{\eta}(\alpha_3 x_1 + x_2) + x_1 + x_2 \cot\theta = 0 \\
&\frac{y'}{y}(\alpha_3 x_1 + x_2) + (2-k)\alpha_2 x_1 = (k-1)\frac{\eta'}{\eta}(\alpha_3 x_1 + x_2)
\end{aligned}\right\} \tag{5}$$

Here, for example, $x_1' = dx_1/d \ln z$.

In case $\alpha_3 = 0$, $z = e^{\theta}$ the system (5) has a solution in the assumed "automodel" form.

If $\alpha_3 \neq 0$, then the desired solutions may occur only for two-dimensional motion of a gas when $x_1 + x_2 \cot\theta$ vanishes, inasmuch as in the opposite case θ clearly enters in the equation the solution of which, according to the hypothesis, must not depend on θ.

Let us investigate first the case $\alpha_3 = 0$.

Here, if we eliminate $\frac{\eta'}{\eta}$, then (5) may be reduced to the form:

$$\left.\begin{aligned}
&\alpha_1 x_1^2 + x_2 x_1' - x_2^2 + (2\alpha_1 + \alpha_2) y = 0 \\
&(1 + \alpha_1) x_1 x_2 + x_2 x_2' + \frac{ky'}{k-1} + \frac{2-k}{k-1}\alpha_2 y \frac{x_1}{x_2} = 0 \\
&(1 + \alpha_1 + \frac{\alpha_2}{k-1}) x_1 + x_2' + \frac{x_2}{k-1}\frac{y'}{y} + x_1 + x_2 \cot\theta = 0
\end{aligned}\right\} \tag{6}$$

Here, for example, $x_1' = dx_1/d\theta$.

Hence it is easy to find Busemann's [1] solution, appearing as generalization of the Prandtl-Meyer [2] solution.

Let us assume that $\alpha_1 = \alpha_2 = 0$. Then all the parameters u, v, p, ρ are functions only of the polar angle θ and equations (6) take the form:

The first equation yields

$$u' = v \tag{7}$$

The second relation is the Bernoulli equation; it yields

$$u^2 + v^2 + \frac{2kw}{k-1} = A^2 = \text{const} \tag{8}$$

After transformation, the third equation becomes

$$(u + u'')\left[\frac{2}{k-1}\,\frac{u'^2}{A^2-(u^2-u'^2)} - 1\right] = u + u' \cot\theta \tag{9}$$

It is not difficult to be convinced that for these motions $S = \text{const}$. In the case of plane motion, the term $u + u' \cot\theta$ vanishes and we arrive at the Prandtl-Meyer solution

$$u' = \sqrt{\frac{k-1}{k+1}\,(A^2-u^2)} \tag{10}$$

In the general case for automodel plane motion (when $\alpha_3 \neq 0$) we have, starting from equations (5), the relations (eliminating η'/η):

$$\left.\begin{aligned}
&\alpha_1 x_1^2 + \alpha_3 x_1 x_1' + x_2 x_1' - x_2^2 + \frac{k}{k-1}\alpha_3 y' + \\
&\qquad (2\alpha_1 + \alpha_2 + \alpha_2\alpha_3\,\frac{2-k}{k-1}\,\frac{x_1}{\alpha_3 x_1 - x_2})\,y = 0 \\
&(1+\alpha_1)x_1 x_2 + \alpha_3 x_1 x_2' + x_2 x_2' + \frac{ky'}{k-1} + \frac{2-k}{k-1}\,\alpha_2\,\frac{x_1 y}{\alpha_3 x_1 + x_2} = 0 \\
&x_1(1 + \alpha_1 + \frac{\alpha_2}{k-1}) + \alpha_3 x_1' + x_2' + \frac{\alpha_3 x_1 + x_2}{k-1}\,\frac{y'}{y} = 0
\end{aligned}\right\} \tag{11}$$

It is evident that the problem must first be reduced to two ordinary differential equations of first degree and then further to one equation of first degree (since the equations are homogeneous).

These equations are

$$\left.\begin{aligned} -\frac{d\ln x}{dx_1} &= \frac{\alpha_3 x_1 + x_2 + \frac{k\alpha_3 y'}{k-1}}{\alpha_1 x_1^2 - x_1^2 + y\left(2\alpha_1 + \alpha_2 + \frac{2-k}{k-1}\,\frac{\alpha_2\alpha_3 k_1}{\alpha_2 x_1 + x_2}\right)} \\ &= \frac{(\alpha_3 x_1 + x_2)x_2' + \frac{ky'}{k-1}}{(1+\alpha_1)x_1x_2 + \frac{2-k}{k-1}\,\frac{\alpha_2 x_1 y}{\alpha_3 x_1 + x_2}} = \frac{\alpha_3 + x_2' + \frac{\alpha_3 x_1 + x_2}{k-1}\,\frac{y'}{y}}{x_1\left(1+\alpha_1 + \frac{\alpha_2}{k-1}\right)} \end{aligned}\right\} \quad 12$$

where

$$x_2' = \frac{dx_2}{d_1}; \quad y' = \frac{dy}{dx_1}.$$

Let us introduce $x_2 = \xi x_1$, $y = \eta x_1^2$; then equations (12) become

$$\frac{d\ln x_1}{d\xi} = \frac{M_1}{N_1} = \frac{M_2}{N_2} \tag{13}$$

where $M_1 = \left[\alpha_1 - \xi^2 + \eta\left(2\alpha_1 + \alpha_2 + \frac{2-k}{k-1}\,\frac{\alpha_2\alpha_3 x_1}{\alpha_3 x_1 + x_2}\right)\right]\left[1 + \frac{\alpha_3 + \xi}{(k+1)\eta}\,\frac{d\eta}{d\xi}\right] -$

$$\frac{k\alpha_3}{k-1}\,\frac{d\eta}{d\xi}\left(1 + \alpha_1 + \frac{\alpha_2}{k-1}\right)$$

$$N_1 = \left(1+\alpha_1 + \frac{\alpha_1}{k-1}\right)\left(\alpha_3 + \xi + \frac{2k\alpha_3\eta}{k-1}\right) -$$

$$\frac{k+1}{k-1}(\alpha_3 + \xi)\left[\alpha_1 - \xi^2 + \eta\left(2\alpha_1 + \alpha_2 + \frac{2-k}{k-1}\,\frac{\alpha_2\alpha_3 x_1}{\alpha_3 x_1 + x_2}\right)\right]$$

$$M_2 = \left[(1+\alpha_1)\xi + \frac{2-k}{k-1}\frac{\alpha_2\eta}{\alpha_3+\xi} \right] \left(1 + \frac{\alpha_3+\xi}{(k-1)\eta}\frac{d\eta}{d\xi}\right) -$$

$$\left(1 + \alpha_1 + \frac{\alpha_2}{k-1}\right)\left(\alpha_3 + \xi + \frac{k}{k-1}\frac{d\eta}{d\xi}\right)$$

$$N_2 = \left(1 + \alpha_1 + \frac{\alpha_2}{k-1}\right)\left[(\alpha_3 + \xi)\,\xi + \frac{2k\eta}{k-1} \right] -$$

$$\frac{k+1}{k-1}(\alpha_3 + \xi)\left[(1+\alpha_1)\xi + \frac{2-k}{k-1}\eta\frac{\alpha_2}{\alpha_3+\xi} \right]$$

Solving (13), we find $\eta = \eta(\xi)$, $x_1 = x_1(\xi)$; moreover, we determine $x_2 = x_2(x_1)$ and $y = y(x_1)$ and then $z = z(x_1)$.

In the case of nonvortical flow, $\frac{\partial u}{\partial \theta} = r\frac{\partial v}{\partial r} + v$ which yields for automodel motion

$$\frac{dx_1}{d\ln z} = \alpha_3\frac{dx_2}{d\ln z} + x_2(1+\alpha_1) \tag{14}$$

For axisymmetric motions ($\alpha_3 = 0$) solving jointly equations (14) and (6) we find that the irrotational flow is only possible for $\alpha_1 = \alpha_2 = 0$; in the same way, the Busemann solution is the unique "automodel" irrotational solution.

The same result occurs for plane motion (in the case of irrotational flow). Comparing the first and second of the equations (11), the coefficients of y and y', we find that $2\alpha_1 + \alpha_2 = 0$.

Here must be:

$$\alpha_1 x_1^2 + \alpha_3 x_1 x_1' + x_2 x_1' - x_2^2 = \alpha_3(1+\alpha_1)x_1 x_2 + \alpha_3 x_2 x_2'(\alpha_3 + 1)$$